Eur. Phys. J. B **6**, 503–510 (1998)

THE EUROPEAN PHYSICAL JOURNAL B

Phase diagram of randomly polymerized membrane

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Received: 3 December 1997 / Revised: 31 March 1998 / Accepted: 3 August 1998

Abstract. Using a replica formalism, a generalization of a recent mean field model corresponding to the observed wrinkling transition in randomly polymerized membranes is presented. In this model we study the effects of global fluctuations of the surface normals to the flat membrane, which can be introduced by a random local field. In absence of these global fluctuations, we show that, the model exhibits both continuous and discontinuous transitions between flat and wrinkled phases, contrary to what has been predicted by Bensimon *et al.* and Attal *et al.* Phase diagrams both in replica symmetry and in breaking of replica symmetry in sense of Almeida and Thouless are given. We have also investigated the effects of global fluctuations on the replica symmetry phase diagram. We show that, the wrinkled phase is favored and the flat phase is unstable. For large global fluctuations, the transition between wrinkled and flat phases becomes first order.

PACS. 87.22.Bt Membrane and subcellular physics and structure – 75.10.Nr Spin-glass and other random models

1 Introduction

Flexible membranes are two dimensional generalization of linear polymer chains. The properties of a 2D membrane, embedded in three dimensional space, depend strongly on the internal order, crystalline, hexatic, or fluid. As in other realizations of 2D matter, defects, and their interactions, affect crucially the stability of a given phase. In contrast to linear polymers, crystalline membranes, also known as tethered or polymerised membranes are expected to exhibit quite different physical properties from their linear counterparts. In particular, they are predicted to have a remarkable low-temperature ordered phase. This ordered, or flat phase, is characterized by long-range order in the orientation of surface normals. At high temperature, or equivalently low bending rigidity, phantom (non-self-avoiding) crystalline membranes will entropically disorder and crumple. Separating these two phases should be a crumpling transition. So, different phases of a crystalline membrane can be distinguished by the behaviour of the surface normals. In the flat phase the normals will have long-range order, while in the crumpled phase the normals eventually decorrelate. These systems have aroused considerable interest both from theoretical and experimental point of view [1].

Theoretically, the crumpling transition of tethered membranes has been understood. Indeed, Nelson *et al.* [2] and Paczuski *et al.* [3] have found that at low temperature a stable flat phase exists. However, more extensive computer simulations [4,5] on large systems have since shown convincingly that, in the presence of only excluded volume interactions, tethered membranes do not crumple but remain flat. This lack of a crumpling transition has been explained in terms of an implicit bending rigidity which is induced by the self-avoidance requirement [6]. An alternative explanation by using a Gaussian variational approximation [7], and an expansion in large embedding space dimension d [8], show that the flat phase is stable for $d = 3$ and 2D membranes crumple only for $d > 4$. This result is in agreement with numerical simulations [5]. In a related calculation, using a variational approach [9], the authors have found that the membrane can crumple only for $d > 3$.

Recent studies of membranes with defects and quenched random disorder have been prompted in part by experimental works [10–12]. These experiments show that partially polymerised membranes undergo, possibly first-order, a reversible phase transition from high-temperature phase characterized by a smooth, floppy surface, to a low-temperature phase. In this last phase the membranes appear rigid and highly wrinkled. It is important to notice that this phase is very different from the so-called crumpled phase of membrane [1], in which the local normals to the membrane fluctuate in time. However, in the wrinkled phase the normals are randomly frozen. Based on the mean field solution of a model for a heterogeneous (disordered) membrane, this transition

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has been linked to the spin glass transition of magnetic systems [11,13–15]. Alternatively, Nelson *et al.* [16] and Morse *et al.* [17] have analysed the stability of the flat phase of disordered membranes by the field theoretical method ($\epsilon = 4 - Dexpansion$), Radzihovsky and Le Doussal [18] have studied this problem in the large d limit. As result of these works it appears that the randomness in the metric destabilizes a flat membrane towards a possible spin glass phase, whereas adding random curvature yield a new $T = 0$ flat phase, with a crossover to a glassy phase at higher temperature.

From the magnetic point of view a membrane model [19] is constrained spin system. In fact, the spin (surface normal) must be normal to the underlying surface, and the constraint is of course essential to the stability of the ordered phase. But within the mean field approximation [2], forfeiting all spatial information, where one is interested in obtaining the possible thermodynamic phases. The constraint may be relaxed as long as it is satisfied in all phases. With this drastic approximation, the disordered membrane model is akin to Heisenberg model with random Dzyaloshinky Moriya (DM) interactions [13,14]. Thus, it might be helpful to keep in mind the magnetic analogy where the ferromagnetic, paramagnetic and spin glass phases are the analog of flat, crumpled and wrinkled phases, respectively.

In this paper we will use a generalized version of that mean-field model of disordered membranes by adding a Gaussian random local field term. This local field can be interpreted as global fluctuations of the surface normals induced by inhomogeneities of elastic properties of membrane. Within the replica symmetry (RS) solution, we have calculated the free energy and constructed set of a self-consistent equations, as is well-known in the mean field procedure. In zero random field, we find that our phase diagram differs in part from that obtained by Bensimon *et al.* [13] and Attal *et al.* [14]. In fact, in our analyzes we show that the model exhibits both a first and second order transitions between flat (ferromagnetic) and wrinkled (spin glass) phases, and reentrant phenomenon occurs. However, in their works [13,14] they predicted within the RS solution that, the line transition between flat and wrinkled phases is only second order. Moreover, by considering small fluctuations of the order parameter around the symmetric solution we derive, analytically, a line of instability. This line is analogous to the Almeida-Thouless (AT) line in spin glass systems [20], above which only a solution involving broken replica symmetry is stable. On the other hand, in presence of random field, we restrict ourselves to the replica symmetric solution. We present a study of the effect of a local quenched random field on the RS phase diagram of randomly frustrated membrane. A phase diagram for various values of random local width is given. We find that the random field plays the same role as that of randomness in preferred metric.

The outline of this paper is as follows: in Section 2 we present the general formalism of replica theory applicable to our model. In Section 3 we derive the RS solution for the order parameters in presence of Gaussian random local field. In Section 4 the RS phase diagram and limit of stability of RS solution against RSB in zero random field are derived, also generalized RS phase diagram is given. Finally, Section 5 contain our conclusions.

2 General formalism of replica theory

We consider the model describing a membrane with quenched random spontaneous curvature [13,14] from the magnetic point of view, to which we added a local quenched random field. Within the mean field approximation [2], the model of a membrane with a randomness, is characterized by the following Hamiltonian

$$H = \frac{-K}{2N} \sum_{\imath \neq \jmath} \mathbf{S}_\imath \mathbf{S}_\jmath - \frac{1}{2} \sum_{\imath \neq \jmath} \mathbf{D}_{\imath\jmath} \left(\mathbf{S}_\imath \wedge \mathbf{S}_\jmath \right) - \sum_\imath h_\imath S_{\imath 1} \quad (1)$$

where $\sum_{\mu=1}^{3} S_{\imath\mu} S_{\imath\mu} = 3$, with $\imath = 1, ..., N$ and the sum $\sum_{\imath \neq \jmath}$ extends over all pairs of pseudospin $\mathbf{S}_\imath$, and K is the bending rigidity. $\mathbf{D}_{\imath\jmath}$ is a random vector attached to the membrane, includes a local random curvature, and $h_\imath$ chosen in $\mu = 1$ direction is a random local field, which can be interpreted as global fluctuations of the normals induced by inhomogeneities of elastic properties of membrane. The $\mathbf{D}_{\imath\jmath}$ and $h_\imath$ are quenched random variables distributed according to their respective Gaussian probability densities:

$$P(\mathbf{D}_{\imath\jmath}) = \left(\frac{2\pi J^2}{N} \right)^{-\frac{3}{2}} \exp\left(-\frac{N}{2} \left(\frac{\mathbf{D}_{\imath\jmath}}{J} \right)^2 \right), \quad (2)$$

$$P(h_\imath) = \left(2\pi h^2 \right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \left(\frac{h_\imath}{h} \right)^2 \right). \quad (3)$$

The N dependence in the probability distribution $P(\mathbf{D}_{\imath\jmath})$ is imposed, as usual for infinite range problems, such as to ensure a sensible thermodynamic limit.

The free energy averaged over the joint probability distribution $P(\mathbf{D}_{\imath\jmath})$ and $P(h_k)$ can be obtained *via* the well-known replica formalism for the partition function Z, *i.e.*:

$$\beta F = - \lim_{n \to 0} \frac{1}{n} [Z^n - 1] \quad (4)$$

or

$$\beta F = - \lim_{n \to 0} \frac{1}{n} \left[\int \prod_{\imath\jmath k} d\, \mathbf{D}_{\imath\jmath} dh_k P(\mathbf{D}_{\imath\jmath}) P(h_k) Tr_n \exp\left[\chi\right] - 1 \right] \quad (5)$$

where

$$\chi = \frac{\beta}{2} \sum_{\imath \neq \jmath} \sum_{\alpha=1}^{n} \left\{ \frac{K}{N} \mathbf{S}_\imath^\alpha \mathbf{S}_\jmath^\alpha + \mathbf{D}_{\imath\jmath} \left(\mathbf{S}_\imath^\alpha \wedge \mathbf{S}_\jmath^\alpha \right) \right\} + \sum_\imath \beta h_\imath S_{\imath 1}^\alpha \quad (6)$$

with $\beta \equiv 1/T$, $\imath$ is indexes the pseudospin and α the replica label. The limit $N \to \infty$ is implied.

Our analysis is a generalization of references [13,14] to the case $h \neq 0$ and that of reference [21] corresponding to the case $J = 0$ and $h \neq 0$ (Gaussian random field Heisenberg model). By carrying out the integrations over $D_{\imath\jmath}$ and h_k in equation (5), and linearizing all quadratic forms which appear by using the standard and non standard Hubbard Stratonovich (HS) transformation [22] and hence saddle point integration, the free energy per pseudospin can be written as, with the same variables used in references [13,14]:

$$\beta f = -\frac{3}{2}\beta_\gamma^2 - \lim_{n\to 0}\frac{1}{n}\min\left[g(m^\alpha, r^{\alpha\beta}, \Delta^{\alpha\beta}, q^\alpha)\right] \quad (7)$$

where

$$g \equiv -\frac{\beta_\gamma k_\gamma}{2}\sum_\alpha m_\alpha^2 + \frac{3}{2}\beta_\gamma^2\sum_\alpha q^\alpha\left(1+q^\alpha\right) - \beta_\gamma^2\sum_{(\alpha\beta)} T_x^{\alpha\beta}\left(T_y^{\alpha\beta} - \frac{T_x^{\alpha\beta}}{4}\right) + \log Tr_n \exp(Q_n) \quad (8)$$

and

$$Q_n \equiv \beta_\gamma k_\gamma\sum_\alpha m_\alpha S_1^\alpha + \frac{1}{2}\sum_\alpha\left((\beta h)^2 - 3\beta_\gamma^2 q^\alpha\right)(S_1^\alpha)^2 + \sum_{(\alpha\beta)}\sum_\nu S_\nu^\alpha S_\nu^\beta\left(\beta_\gamma^2 T_\nu^{\alpha\beta} + (\beta h)^2\delta_{\nu,1}\right) \quad (9)$$

above, α and β are replica labels and $\sum_{(\alpha\beta)}$ denote sums over distinct pairs of replica. The notations $\beta_\gamma \equiv \beta J$, $k_\gamma \equiv K/J$,$T_\mu^{\alpha\beta} \equiv 2r^{\alpha\beta} - (3\delta_{\mu,1} - 1)\Delta_{\alpha\beta}$ has been used.

The extrema of the functional $g(m^\alpha, r^{\alpha\beta}, \Delta^{\alpha\beta}, q^\alpha)$ give us the coupled self-consistent equations:

$$\begin{aligned} m^\alpha &= \langle S_1^\alpha\rangle,\ \forall\alpha \\ r^{\alpha\beta} &= \frac{1}{3}\sum_{\mu=1}^{3}\langle S_\mu^\alpha S_\mu^\beta\rangle,\ \forall(\alpha\beta) \\ \Delta^{\alpha\beta} &= \frac{1}{6}\sum_{\mu=1}^{3}(3-\delta_{\mu,1})\langle S_\mu^\alpha S_\mu^\beta\rangle,\ \forall(\alpha\beta) \\ 1+2q^\alpha &= \langle (S_1^\alpha)^2\rangle,\ \forall\alpha \end{aligned} \quad (10)$$

where $\langle\dots\rangle$ indicate thermal averages with respect to the effective Hamiltonian Q_n. The parameters m^α, $r^{\alpha\beta}$, $\Delta^{\alpha\beta}$ and q^α are the magnetization, isotropic, anisotropic part of spin glass and quadrupolar order parameters, respectively. As in other anisotropic vector models [23] the order parameters q^α and m^α are strictly α-independent both in replica symmetric state and in the case of replica symmetric breaking (RSB).

3 Replica symmetric solution

The RS state is obtained by setting $r^{\alpha\beta} = r$, $\Delta^{\alpha\beta} = \Delta$ and $T_\mu^{\alpha\beta} = T_\mu$ in equations (8, 9). Then, the analytic continuation $n \to 0$ in equation (7) may be easily performed. With this choice, the free energy per pseudospin is given by:

$$\beta f = \frac{\beta_\gamma k_\gamma}{2}m^2 - \frac{3}{2}\beta_\gamma^2\left(1+q+q^2\right)\frac{\beta_\gamma^2}{2}T_1\left(T_2 - \frac{T_1}{4}\right) + \frac{3}{2}\beta_\gamma^2 T_1 - \int\prod_{\mu=1}^{3}\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}{t_\mu}^2\right)\log(Z) \quad (11)$$

with

$$Z = \int_{|\mathbf{S}|=\sqrt{3}} d\mathbf{S}\exp\left(\sum_{\mu=1}^{3} a_\mu S_\mu + \frac{b}{2}S_1^2\right) \quad (12)$$

where

$$a_\mu = t_\mu\left(\beta_\gamma^2 T_\mu + (\beta h)^2\delta_{\mu,1}\right)^{\frac{1}{2}} + \beta_\gamma k_\gamma m\delta_{\mu,1} \quad (13)$$

and

$$b = 3{\beta_\gamma}^2(\Delta - q). \quad (14)$$

By linearizing the quadratic forms in S_1, the trace in equation (12) can be evaluated as an integral over the 3-dimensional solid angle. We finally have:

$$Z = 3\int\frac{dw}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}w^2\right)\frac{\sinh\left(\sqrt{3}A\right)}{\sqrt{3}A} \quad (15)$$

with

$$A = \left(\sum_{\mu=1}^{3}\left(a_\mu + \delta_{\mu,1}\sqrt{b}w\right)^2\right)^{\frac{1}{2}}. \quad (16)$$

Thus, the set of coupled self consistent equations now yields:

$$\begin{aligned} m &= \int\prod_\mu\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t_\mu^2\right)\left(\frac{1}{Z}\frac{\partial Z}{\partial a_1}\right) \\ r &= \frac{1}{3}\int\prod_\mu\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t_\mu^2\right)\sum_{\mu=1}^{3}\left(\frac{1}{Z}\frac{\partial Z}{\partial a_\mu}\right)^2 \\ \Delta &= \frac{1}{6}\int\prod_\mu\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t_\mu^2\right)\sum_{\mu=1}^{3}(3-\delta_{\mu,1})\left(\frac{1}{Z}\frac{\partial Z}{\partial a_\mu}\right)^2 \\ 1+2q &= \int\prod_\mu\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t_\mu^2\right)\left(\frac{1}{Z}\frac{\partial^2 Z}{\partial^2 a_1}\right). \end{aligned} \quad (17)$$

We could have also obtained the set of equation (17) directly from equation (10), since within the RS approximation any disorder averaged product of thermodynamic averages is simply related to an average in replica space [24].

4 Results and discussions

4.1 Phase diagram in zero random field

In this section we present numerical solutions to the set of coupled consistent equations, equation (17) for zero width of random field. First, it is well-known that in the absence of an homogeneous field, the quadrupolar parameter q is zero, while the anisotropic part of the spin glass order parameter vanishes in spin glass and paramagnetic phases [25] (see below, Fig. 2). Consequently, the phase diagram can be determined with $\Delta = q = 0$. The resulting equations are given by:

$$m=\int\prod_{\mu}\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t_\mu^2\right)\left[\frac{\sqrt{3}a_1}{|a|}\coth\left(\sqrt{3}|a|\right)-\frac{a_1}{|a|^2}\right]$$
$$r=\frac{1}{3}\int\prod_{\mu}\frac{dt_\mu}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t_\mu^2\right)\left[\sqrt{3}\coth\left(\sqrt{3}|a|\right)-\frac{1}{|a|}\right]^2 \tag{18}$$

where

$$|a| = \left(\sum_{\mu=1}^{3} a_\mu^2\right)^{\frac{1}{2}} \tag{19}$$

and

$$a_\mu = t_\mu \beta_\gamma \sqrt{2r} + \beta_\gamma k_\gamma m \delta_{\mu,1}. \tag{20}$$

Hence, only three phases are possible [25] in this case, namely, the paramagnetic ($r = 0, m = 0$), the isotropic spin glass ($r \neq 0, m = 0$) and ferromagnetic ($r \neq 0, m \neq 0$), which correspond to, crumpled, wrinkled and flat phases of membrane, respectively.

At this level, we can examine the equation (18) analytically. If the transition line from wrinkled phase to flat phase is continuous, we can expand the equation (18) for small m. Then the second-order transition line is given by:

$$k_\gamma^{-1} = -\frac{1}{6\beta_\gamma\left(1-(\beta_\gamma k_\gamma)^{-1}\right)} + \frac{1}{\sqrt{3\pi}\left(1-(\beta_\gamma k_\gamma)^{-1}\right)^{\frac{1}{2}}} \times \int_0^\infty dt\exp\left(-\frac{1}{2}t^2\right)t^3\coth\left(\sqrt{6}\beta_\gamma\left(1-(\beta_\gamma k_\gamma)^{-1}\right)^{\frac{1}{2}}t\right) \tag{21}$$

the critical value of disorder at $T = 0$ for which the flat phase becomes unstable is $J_c/K = \sqrt{4/3\pi}$, which is far from the value $2/\sqrt{18\pi}$, as quoted by previous authors [13,14].

Moreover, by performing an expansion in powers of a_μ up to eighth order, which is equivalent to high temperature expansion for βf up to seventh order in β_γ, we find from equation (18):

$$m=\beta_\gamma k_\gamma m-2{\beta_\gamma}^2(\beta_\gamma k_\gamma m)r-\frac{1}{5}(\beta_\gamma k_\gamma m)^3+8{\beta_\gamma}^4(\beta_\gamma k_\gamma m)r^2 +\frac{8}{5}{\beta_\gamma}^2(\beta_\gamma k_\gamma m)^3r+\frac{2}{35}(\beta_\gamma k_\gamma m)^5-\frac{216}{5}{\beta_\gamma}^6(\beta_\gamma k_\gamma m)r^3 -\frac{324}{25}{\beta_\gamma}^4(\beta_\gamma k_\gamma m)^3r^2-\frac{162}{175}{\beta_\gamma}^2(\beta_\gamma k_\gamma m)^5r-\frac{3}{175}(\beta_\gamma k_\gamma m)^7$$

$$r=2{\beta_\gamma}^2r+\frac{1}{3}(\beta_\gamma k_\gamma m)^2-8{\beta_\gamma}^4r^2-\frac{8}{3}{\beta_\gamma}^2(\beta_\gamma k_\gamma m)^2r -\frac{2}{15}(\beta_\gamma k_\gamma m)^4+\frac{216}{5}{\beta_\gamma}^6r^3+\frac{108}{5}{\beta_\gamma}^4(\beta_\gamma k_\gamma m)^2r^2 +\frac{54}{25}{\beta_\gamma}^2(\beta_\gamma k_\gamma m)^4r+\frac{9}{175}(\beta_\gamma k_\gamma m)^6. \tag{22}$$

Equation (22) indicates that the model exhibits a second order transition between flat and crumpled phases at $T/K = 1$ for $J/K < \sqrt{2}/2$, and between crumpled and wrinkled phases at $J/K = \sqrt{2}/2$ for $T/K > 1$. Indeed, this equation provide information only for high temperature and is not able to reproduce the correct phase diagram at low temperature. However, the RS phase diagram can be obtained completely by investigating the detailed numerical solutions of equation (17) (with $h = 0$) each time with a minimization of the free energy given by equation (11) to locate the correct transition point.

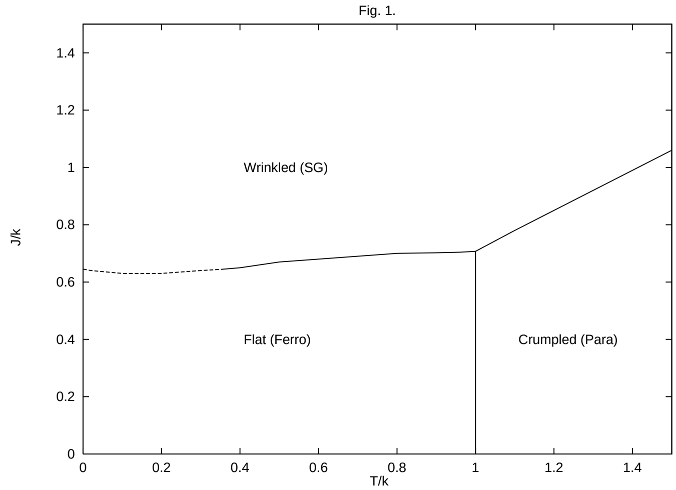

Fig. 1. Phase diagram of Hamiltonian (1) with zero random local field h/K. The continuous line is the second order transition and the dashed line is the first order transition.

The result is shown in Figure 1 where we plot the RS phase diagram supported by the behavior of order parameters m, r, Δ and q as a function of temperature. The existence of crumpled ($m = 0, r = 0$), wrinkled ($m = 0, r \neq 0$) and flat ($m \neq 0, r \neq 0$) phases were previously recognized [13,14]. The novel features of the phase diagram is the reentrance phenomenon which appears at low temperature, and the existence of first and second order transition between wrinkled and flat phases.

As seen in Figure 2, one observes that for $J/K = 0.4$ and $J/K = 0.9$, there are continuous transitions respectively, between flat and wrinkled phases (Fig. 2a),

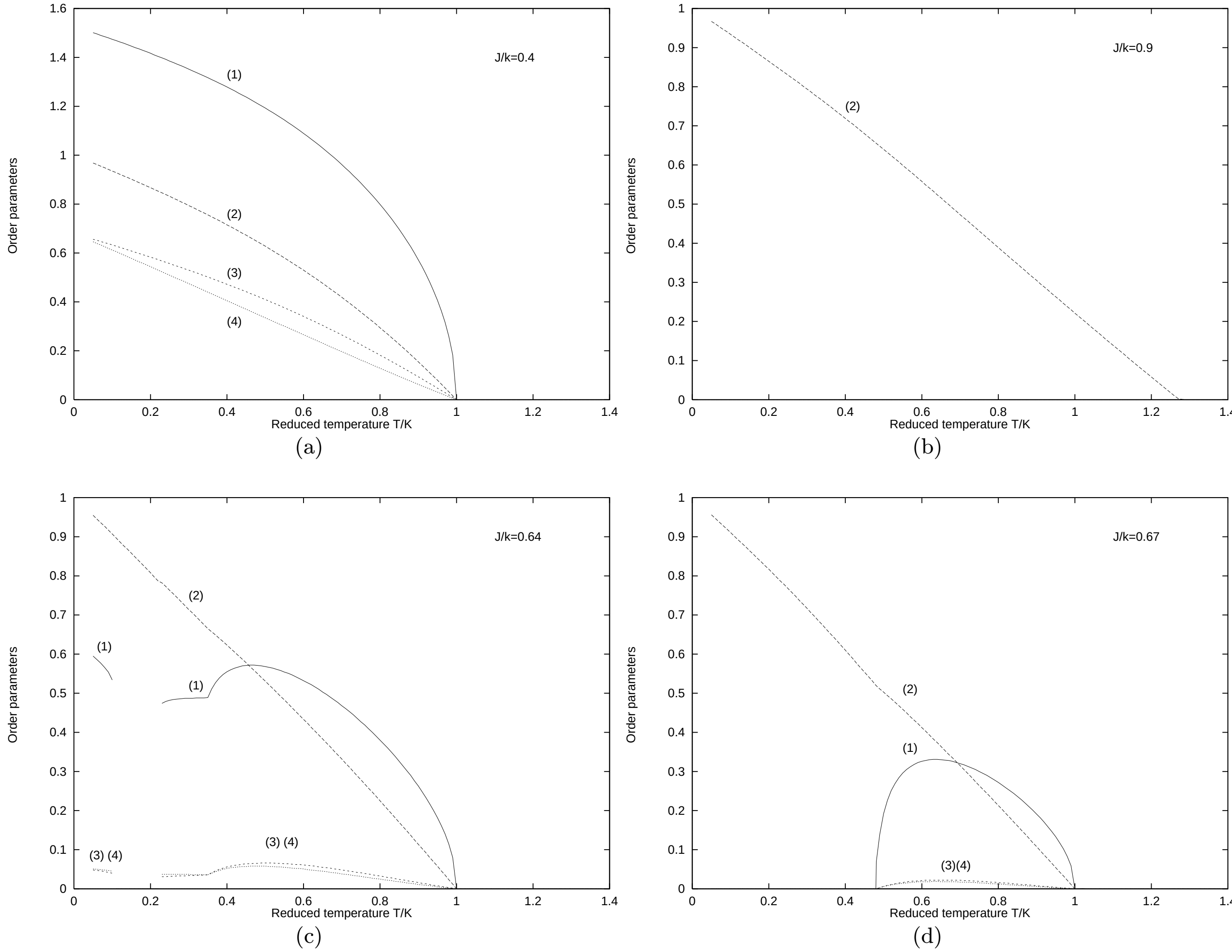

Fig. 2. Order parameters as a function of reduced temperature for $h/K = 0$ and various values of J/K. The label (1), (2), (3) and (4) correspond respectively to m, r, Δ and q.

and between wrinkled and crumpled phases (Fig. 2b) (see also Eq. (22)). Whereas, in Figure 2c, we show that the model exhibits a reentrant transition (flat-wrinkled-flat) at low temperature for $J/K = 0.64$, by a first order transition from a flat to wrinkled phase and from a wrinkled to flat phase; while for $J/K = 0.67$ this transition is of second order (Fig. 2d). Note that in Figure 2, the variation of order parameters q and Δ as function of temperature is very small in vicinity of the flat-wrinkled phase transition, then our assumption $\Delta = q = 0$ is reasonable. Consequently, the second order boundary between flat and wrinkled phases can be obtained analytically by equation (21).

4.2 Stability limit of RS solution in zero random field

It is well-known that the RS solution is generally unstable against RSB [20,26]. In presence or absence of an homogeneous external field, a borderline separating the regions of stable and unstable RS solution can be drawn, known as the AT line [20]. Thus the spin glass transition should correspond to a change from a single RS spin glass order parameter at $T > T_g$ (where T_g is the temperature of freezing) to a RSB form at $T < T_g$ represented by the Parisi function $r(x)$ [27], $0 \leq x \leq 1$. Following Almeida and Thouless the stability of given solution is ensured by positive definite Hessian matrix associated with the functional $g(m^\alpha, r^{\alpha\beta}, \Delta^{\alpha\beta}, q^\alpha)$ given by equation (8).

As discussed in detail in references [20,26], the problem of stability reduces to the requirement that all eigenvalues of this Hessian matrix must be negative. In our case, it can be shown that the RS solution is stable if:

$$\beta_\gamma^{-2} > \frac{2}{3} \int \prod_\mu \frac{dt_\mu}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t_\mu^2\right) \phi(m, r) \qquad (23)$$

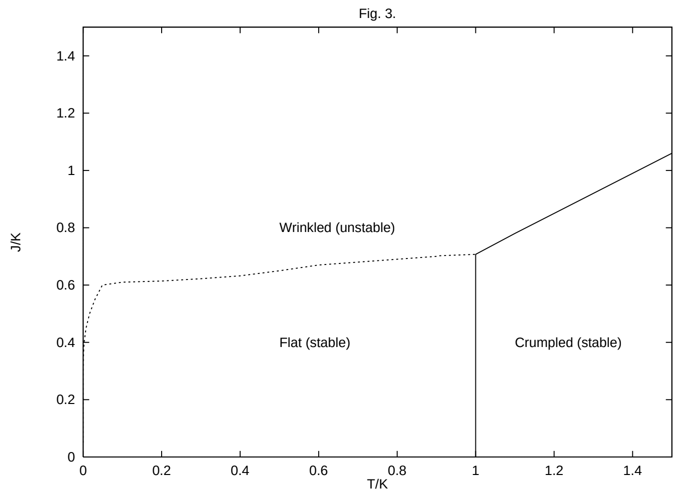

Fig. 3. Phase diagram showing the limits of unstability of replica symmetry solution.

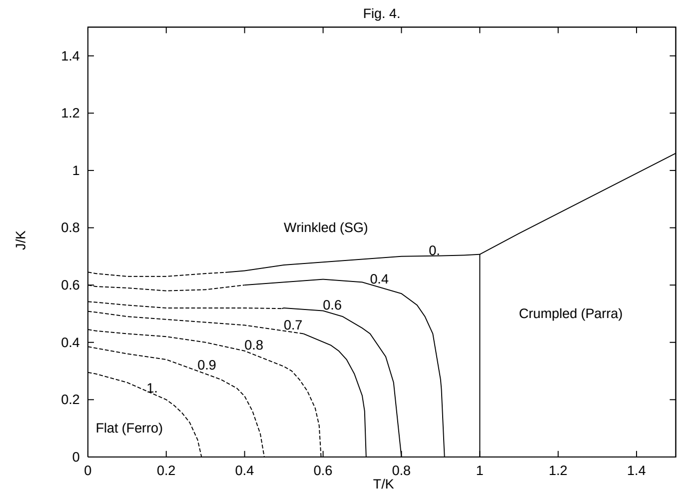

Fig. 4. Mean field phase diagram in the presence of a random local field of width h/K. The number accompanying each curve correspond to the selected values of h/K. The dashed lines define the first order transition and the solid line denote the second order transition.

where

$$\phi(m,r) \equiv 9\coth^4\left(\sqrt{3}|a|\right) + 9\left(1 - 2\coth^2\left(\sqrt{3}|a|\right)\right) - \frac{4\sqrt{3}}{|a|^3}\coth\left(\sqrt{3}|a|\right) + \frac{3}{|a|^2}\left(2 + \frac{1}{|a|^2}\right) \tag{24}$$

the order parameter m and r are given by equation (18).

The line of instability obtained by numerical evaluation of simultaneous solution of equation (18) and the equality in equation (23) is shown in Figure 3. Above that line only a solution with broken replica symmetry provides a correct description of wrinkled state. It should be noted that approximate analytic expression for this line can be derived for the region where T and J go to zero. The result we have in this region is:

$$\frac{T}{J} \sim \alpha_0 \exp\left(-\frac{3}{4}\left(\frac{K}{J}\right)^2\right) \tag{25}$$

where $\alpha_0 = 0.65416$. Notice that the AT line of this model has the same behavior at low temperature as the AT line of a SK model. This result is not surprising because of the strong coupling between the spin components in this model [25].

4.3 Effect of random field on flat phase

Recently it has been shown that randomness in the metric destabilizes a flat membrane towards a wrinkled phase [18,28], whereas random curvature yields a new $T = 0$ flat phase [13,14]. Here we discuss the effect of random local field on the flat phase. Within the replica symmetric solution, the free energy per pseudospin and the equilibrium equations are given by equations (11, 17), respectively. As is seen in equation (11), the variance of the random field h acts as an effective ordering field for the wrinkled state ($r \neq 0, m = 0$), the corresponding order parameters r and Δ are non zero at all temperatures. That behavior which have already been presented in the literature [29,30], can be seen from the high temperature expansion of the equation (17) which gives at relevant order to:

$$\begin{aligned} m &= (\beta_\gamma k_\gamma m)\left(1 - \frac{6}{5}(\beta h)^2\right) \\ &\quad - \frac{1}{5}(\beta_\gamma k_\gamma m)^3\left(1 - \frac{40}{7}(\beta h)^2\right) + \cdots \\ r &= \frac{2}{3}(\beta h)^2 + 2{\beta_\gamma}^2 r\left(1 - \frac{8}{3}(\beta h)^2\right) + \cdots \\ \Delta &= \frac{2}{3}(\beta h)^2 - 2{\beta_\gamma}^2 \Delta\left(1 - \frac{68}{15}(\beta h)^2\right) + \cdots \\ q &= \frac{2}{5}(\beta h)^2 - \frac{3}{5}{\beta_\gamma}^2 q\left(1 + \frac{4}{7}(\beta h)^2\right) + \cdots \end{aligned} \tag{26}$$

Thus, a conventional phase transition, where r would change from zero to a non zero value, is here remarkably absent (see Fig. 5).

In order to draw a phase diagram for different values of h/K, we have studied the detailed numerical solutions of equation (17) with a minimization of the free energy given by equation (11). The surface of second order phase transitions can be calculated analytically by linearizing the parameter m in equation (17) around $m = 0$, which gives:

$$\frac{T}{K} = [1 + 2q - (r + 2\Delta)] \tag{27}$$

where r, Δ and q are determined *via* equation (17). The resulting phase diagram is displayed in Figure 4. From which, one can observes that there exists a tricritical line separating the surfaces of second and first order transitions.

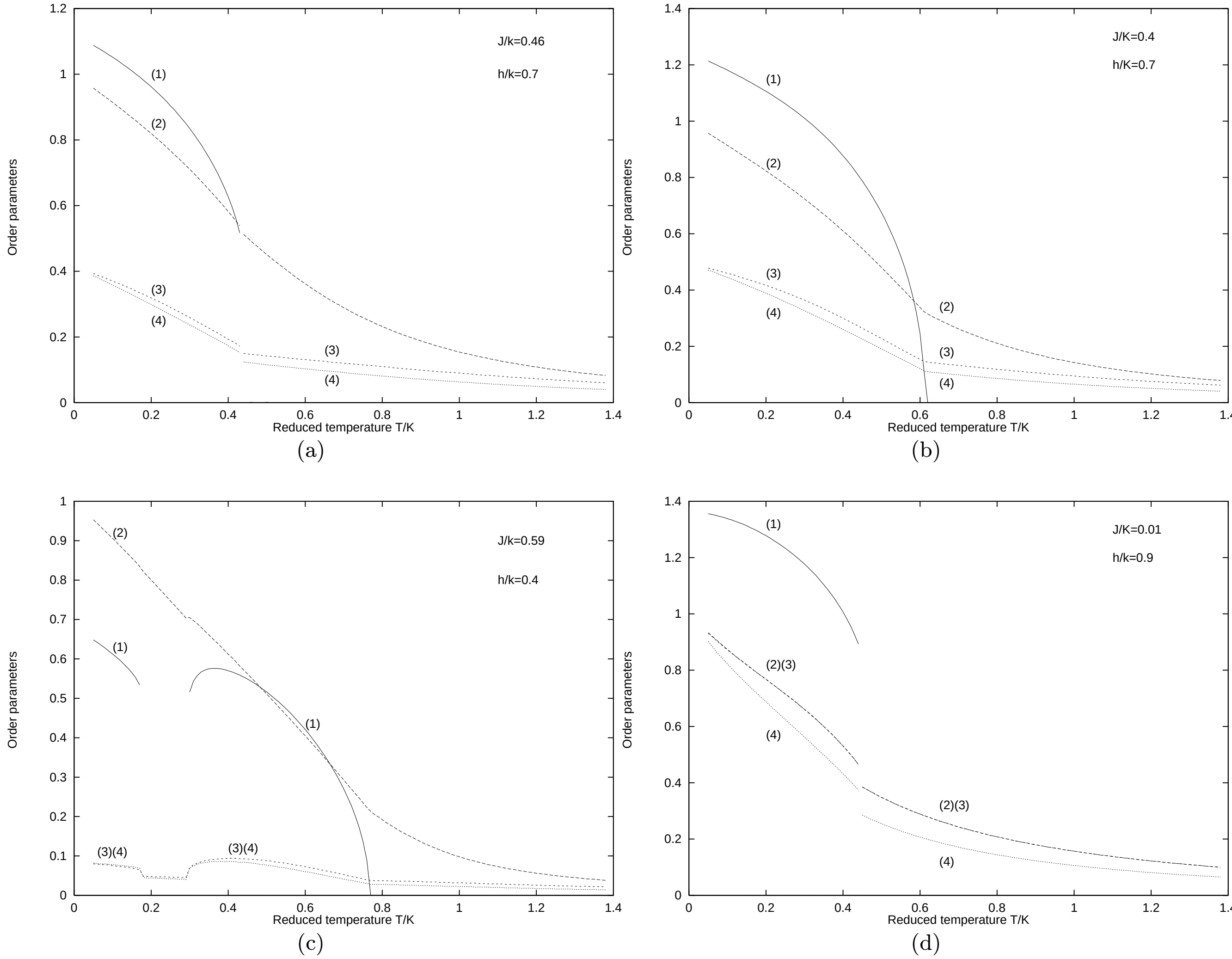

Fig. 5. Order parameters as function of reduced temperatures for selected values of h/K and J/K. The label (1), (2), (3) and (4) correspond respectively to m, r, Δ and q.

The dependence of m, r, Δ and q as function of the temperature for some relevant values of h/K and J/K, is shown in Figure 5. From these figures it is clear that the magnetization undergoes a discontinuity at the first order transition (Fig. 5a), while its vanishes continuously at the second order transition (Fig. 5b). For $h/K = 0.4$ and $J/K = 0.59$, the magnetization vanishes twice (Fig. 5c); this is a feature of a reentrant phenomena in agreement with the phase diagram in Figure 4. However, when increasing the variance of random field h/K, the reentrant phenomena is canceled. This result is in agreement with previous works related to spin glass model [30]. Moreover, by increasing the variance of random field, the flat phase is unstable and the wrinkled phase takes over this result is consistent with what has been concluded in references [18,28]. Furthermore, when h/K become large then $h/K = 0.712$ (see Fig. 5d), the random field makes the transition between flat and wrinkled phase first order, which is in agreement with our previous investigation of the Gaussian random field Heisenberg model using the replica technique [21].

5 Conclusion

We have studied the randomly frustrated membrane model in presence of a Gaussian random field with variance h. This type of randomness could be caused by a global fluctuations of surface normals to the membrane. Within replica symmetry and $h = 0$, we have take up again the investigation of randomly disordered membrane model which has been introduced previously [13,14]. We find that, the model exhibits a second and first order transition between flat (ferromagnetic) and wrinkled (spin glass) phases, and the reentrance phenomenon occurs at low temperature. These main results which are not mentioned in references [13,14]. A related model has been studied by Rubinstein *et al.* [31] who also pointed out the existence of a reentrant transition from a ferromagnetic (flat) ordered phase to a spin glass (wrinkled) phase. However, by analogy to spin glass problem [30], the reentrance of wrinkled phase is shown to be linked to the negative entropy at $T = 0$ which makes the RS solution to be a poor representation of the wrinkled phase. Thus, it would be very useful to find a replica symmetry breaking solution of a such model with $h = 0$. We note that a Parisi RSB formalism for this model has been derived by Attal *et al.* [14]. However, in our analysis of RSB we have follow the same route as Almeida and Thouless in their investigation of instability limit of RS solution for S–K model [20]. We have find the analytical equations of the equivalent AT line of that model. Therefore, as in reference [14], within the RSB scheme we can concluded that (see Eq. (25)) the model exhibits a wrinkled-flat phase as T go to zero even for infinitesimal disorder. On other hand, we have also investigate the randomly frustrated membrane by adding a Gaussian random field, within the RS solution. We have derive a generalized phase diagram of the model discussed here and that in reference [21]. We find that the random field width makes the flat phase unstable and the wrinkled phase takes over. Moreover, the reentrant effect is completely removed when h increases and the transition between wrinkled and flat phase is first order.

This work has been made in the frame of an exchange program between the CNRI (Italy) and the CNCPRST (Morocco).The authors would like to thank both organization.

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